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Test of finite-size scaling predictions in three dimensions

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Abstract. A constrained monomer-dimer (CMD) model on decorated hypercubic lattices, which is exactly solvable in any dimensionality d, is studied. The model is critical in the limit of maximal packing of the dimer configurations allowed by the constraint. For $d=2$ it is equivalent to the rooted-tree model on the square lattice considered by Duplantier and David. The validity of two finite-size scaling predictions in higher dimensionalities is studied: (i) the existence of logarithmic corrections in the free energy due to corners, and (ii) the amplitude-exponent relation for the pair correlation function. The logarithmic finite-size corrections are obtained for arbitrary d, for boundary conditions which are periodic for $d' \geqslant 0$ dimensions and free in the remaining $d-d'$ dimensions. Dimer-dimer correlation functions are studied at $d=3$ for a system finite in two dimensions and infinite in the third. It is shown that the amplitude-exponent relation for the CMD model holds for $d=2$ and breaks down for $d=3$.

1. Introduction

There are two important finite-size predictions of the conformal theory of two-dimensional systems at criticality [1] (for a review see [2]) which have been generalized to higher dimensionalities: these are the logarithmic corrections to the free energy arising from corners [3], and the amplitude-exponent relation [4].

In the first case one considers a fully finite system of characteristic size L with non-smooth boundaries and free (non-periodic) boundary conditions. As was shown by Cardy and Peschel [3], the free energy of two-dimensional conformally invariant models on manifolds with Euler number $\chi = 0$ contains logarithmic finite-size corrections, ΔF^{corner}, arising from each corner with interior angle γ,

$$\Delta F^{\text{corner}} = \frac{c\gamma}{24\pi}\left[1-(\pi/\gamma)^2\right]\ln L \tag{1.1}$$

where c is the conformal anomaly number. By using finite-size scaling arguments Privman [5] has predicted similar corrections for systems of arbitrary dimensionality d,

$$\Delta F^{(c)} = \sum_{\text{corners}} y_i \ln L \tag{1.2}$$

with some universal amplitudes y_i attributed to each corner.

In the second case one considers a system infinite in one dimension and finite in the other dimensions. For a two-dimensional infinitely long strip of width L, the

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correlation functions of a scaling operator ϕ_i decay exponentially along the strip with a correlation length $\xi_i \propto L$. Cardy [4] has shown that, as a consequence of conformal invariance, the proportionality coefficient turns out to be simply related to the exponent $2x_i$ of the algebraic decay of the two-point correlation function for the corresponding critical infinite-plane system:

$$\xi_i^{-1} = A_i L^{-1} \qquad A_i = 2\pi x_i. \tag{1.3}$$

This observation motivated Henkel [6] to ask whether the linear relation between the bulk critical exponent x and the finite-size scaling amplitude A holds in three dimensions.

The extension to higher dimensionality of both of the predictions (1.2) and (1.3) has been checked by the example of a few models.

Logarithmic contributions in the free energy, $\Delta F^{(l)}$, at arbitrary dimensionality and three different types of boundary conditions, have been obtained in a Gaussian-type model by Gelfand and Fisher [7]; these contributions were found to arise in the small block limit due to the zero-eigenvalue mode. Here we quote the corresponding results for a massive Gaussian model on a d-dimensional hypercubic lattice of unit spacing, when the linear size $L \to \infty$ and the mass $m \to 0$, so that $mL \to 0$: (i) for *free* boundary conditions

$$\Delta F_{\mathrm{G}}^{(l),\mathrm{free}} = (1-2^{-d}) \ln(mL) \tag{1.4}$$

(ii) for *fixed* boundary conditions

$$\Delta F_{\mathrm{G}}^{(l),\mathrm{fixed}} = (-1)^{d+1} 2^{-d} \ln(mL) \tag{1.5}$$

(iii) for *periodic* boundary conditions

$$\Delta F_{\mathrm{G}}^{(l),\mathrm{periodic}} = \ln(mL). \tag{1.6}$$

Note that both conformal theory [3] and finite-size scaling arguments [5] predict that logarithmic corner corrections to the free energy density should be absent for periodic boundary conditions. However, such terms have been found by Duplantier and David [8] in the two-dimensional conformally invariant spanning tree (ST) model both under *free*,

$$\Delta F_{\mathrm{ST}}^{(l),\mathrm{free}} = -\tfrac{3}{2} \ln L \qquad (d=2) \tag{1.7}$$

and *periodic*

$$\Delta F_{\mathrm{ST}}^{(l),\mathrm{periodic}} = -2 \ln L \qquad (d=2) \tag{1.8}$$

boundary conditions.

Finally, we mention that the logarithmic corner corrections in the three-dimensional Ising model with free boundary conditions have been studied numerically by Lai and Mon [9]. They have found that the Monte Carlo data for the contribution to the free energy from the corners has the form

$$\Delta F_{\mathrm{I}}^{(c),\mathrm{free}} = u \ln L + \mathrm{const} \tag{1.9}$$

and obtained the following estimates for the universal amplitude:

$$u = \begin{cases} 0.009 \pm 0.005 & \text{for the simple cubic lattice} \\ 0.012 \pm 0.003 & \text{for the body-centred cubic lattice.} \end{cases} \tag{1.10}$$

As far as the amplitude–exponent relation (1.3) in higher dimensions is concerned, Henkel [6] studied the critical (2+1)-dimensional Ising model infinite in one direction but finite in the other two directions, with antiperiodic boundary conditions. He obtained a numerical evidence for the relation

$$A_\varepsilon / A_\sigma = x_\varepsilon / x_\sigma \tag{1.11}$$

where the subscripts σ and ε denote the spin–spin and energy–energy correlations, respectively. Somewhat later, Henkel [10] considered the critical spherical model in the same geometry and proved that the equality (1.11) is exact both for periodic and antiperiodic boundary conditions.

The aim of this work is to test the predictions (1.2) and (1.3) in the case of another exactly solvable three-dimensional model, namely the constrained monomer–dimer (CMD) model [11].

The main results may be summarized as follows:

(i) The difference in the free energies $F_{d,0} - F_{d,d'}$ of the d-dimensional CMD model, with $F_{d,0}$ corresponding to fully free boundary conditions, and $F_{d,d'}$ corresponding to periodic boundaries in $d' \geqslant 1$ dimensions and free boundaries in the remaining $d - d'$ dimensions, contains the logarithmic correction term $2^{1-d} \ln L$ which comes from the 2^d corners of the system with linear size L.

(ii) A comparison of two different dimer–dimer correlation functions of the CMD model shows that the amplitude–exponent relation holds in $d = 2$ dimensions and breaks down in $d = 3$ dimensions for both periodic and antiperiodic boundary conditions.

Section 2 contains the formulation of the model. In section 3 the logarithmic finite-size corrections under different boundary conditions are derived. Section 4 is devoted to the test of the amplitude–exponent relations. A short discussion is given in the final section.

2. The model

To formulate the model, consider the d-dimensional simple hypercubic lattice $\mathscr{L}_d$. Put an additional site at the middle of each lattice bond. Thus, one gets a lattice $\mathscr{L}_d^*$, a two-dimensional analogue of which, $\mathscr{L}_2^*$, is shown in figure 1.

The monomer–dimer problem on $\mathscr{L}_d^*$ consists of enumerating all possible arrangements of non-intersecting dimers, i.e. of two-atomic molecules occupying two neighbouring sites of the lattice (see figure 1). We require also that any closed contour on

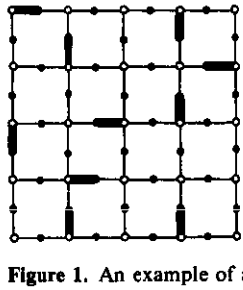

Figure 1. An example of an allowed dimer configuration for $d = 2$. Sites of the quadratic lattice $\mathscr{L}_2$ are shown by open circles, the decorated lattice $\mathscr{L}_2^*$ consists of open and solid circles; dimers are represented by black bars.

$\mathscr{L}_d^*$ passing through $2n$ sites should not pass along more than $(n-1)$ dimers. We call the resulting model the CMD problem [11]. The partition function of the model is

$$\Lambda(z) = \sum_{k=0}^{N_d} z^k g(k) \tag{2.1}$$

where $g(k)$ is the number of allowed arrangements of k dimers on the given lattice and N_d is the number of sites of the lattice $\mathscr{L}_d$. It has been shown in [12] that the above-formulated model is equivalent to the rooted-tree model which can be solved in arbitrary dimensions by the Kirhhoff theorem (see [8] for references).

To establish this equivalence, it is convenient to regard a collection of dimers on the lattice $\mathscr{L}_d^*$ as a set of arrows on the lattice $\mathscr{L}_d$. An arrow is attached to a site s of $\mathscr{L}_d$ if the corresponding site of $\mathscr{L}_d$ is occupied by a dimer. An arrow occupying a site s is directed from s along the bond of $\mathscr{L}_d$ occupied by a dimer to the nearest neighbour s'. We say that the arrow generates a path ss' from s to s'. A collection of paths of the form $s_1s_2, s_2s_3, \ldots, s_{n-1}s_n$, generated by arrows at the sites $s_1, s_2, \ldots, s_{n-1}$, is a path from s_1 to s_n. If the site s_n coincides with the site s_1, the path s_1s_n is closed. If there is no arrow at the site s_n, this site is the end-point of the path s_1s_n.

A set of configurations of n arrows generating no closed paths is in one-to-one correspondence with the set of allowed dimer configurations on $\mathscr{L}_d^*$. There is no difficulty in showing that each configuration of arrows corresponds to one and only one configuration of rooted trees. Indeed, consider an arbitrary configuration of arrows which generates a collection of paths ending at sites $s_1, s_2, \ldots, s_n$. Then a collection of bonds belonging to all paths that end at a site s_j forms a tree having the root s_j. Since any two paths ending at different sites have no common elements, each tree has exactly one root.

Conversely, let $\mathscr{C}$ be a configuration of trees having the roots $s_1, s_2, \ldots, s_n$. Consider a tree whose root is s_j. Ascribe to each vertex s of the tree an arrow directed from s to the nearest neighbour s' for which the distance (the number of connected bonds) between s' and s_j is minimal. Repeating this construction for every tree we get an arrow configuration. Thus, the required one-to-one correspondence between the rooted-tree model and the CMD model is established.

Let us return to the original cubic lattice $\mathscr{L}_d$ and introduce an adjacency matrix $\mathbf{C}$ as the matrix, whose elements $C_{n,m}$ are labelled by the vertices of $\mathscr{L}_d$ and are defined as follows:

$$C_{n,m} = \begin{cases} -z & \text{if } n \neq m \text{ and } n, m \text{ are nearest neighbours} \\ 2dz & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

The solution of the CMD problem is [11]

$$\Lambda(z) = \det(\mathbf{C} + \mathbf{I}). \tag{2.3}$$

For a lattice of the block geometry $L_1 \times \ldots \times L_d$, with periodic, antiperiodic or free boundary conditions, equation (2.3) has the explicit form

$$\Lambda(z) = \prod_{\nu=1}^{d} \prod_{k_\nu=0}^{L_\nu - 1} [1 + z\lambda_d(k)] \tag{2.4}$$

where the eigenvalues $\lambda_d(k)$ of the matrix $\mathbf{A} = \mathbf{C}/z$ depend on the boundary conditions.

Thus, for fully periodic boundaries one has

$$\lambda_d(\boldsymbol{k}) = 2d - 2\sum_{\nu=1}^{d} \cos\frac{2\pi k_\nu}{L_\nu}. \tag{2.5}$$

We will need also the explicit expressions for the eigenvalues of $\mathbf{C}/z$ under boundary conditions which are periodic in d' dimensions and free in the remaining $d-d'$ dimensions,

$$\lambda_{d,d'}(\boldsymbol{k}) = 2d - 2\sum_{\nu=1}^{d'} \cos\frac{2\pi k_\nu}{L_\nu} - 2\sum_{\nu=d'+1}^{d} \cos\frac{\pi k_\nu}{L_\nu} \tag{2.6}$$

or periodic in d' dimensions and antiperiodic in the remaining $d-d'$ dimensions,

$$\bar{\lambda}_{d,d'}(\boldsymbol{k}) = 2d - 2\sum_{\nu=1}^{d} \cos\frac{2\pi k_\nu}{L_\nu} - 2\sum_{\nu=d'+1}^{d} \cos\frac{\pi(2k_\nu+1)}{L_\nu}. \tag{2.7}$$

Duplantier and David [8] have investigated an equivalent model in two dimensions and argued that the model can be reduced to massive two-dimensional free field theory, critical at zero mass (infinite fugacity z).

The crucial property of the CMD model in two and higher dimensions is the behaviour of the correlation functions which decay algebraically at the critical point and exponentially elsewhere.

Below, we shall consider the critical properties of the CMD model to obtain the logarithmic corrections to the free energy and to check the amplitude-exponent relation for periodic and antiperiodic boundary conditions.

3. Logarithmic corrections

At the critical point $z \to \infty$, the partition function (2.4) can be written in the form

$$\Lambda(z) = (z)^{N_d}\Lambda \tag{3.1}$$

where

$$\Lambda = \prod_{\nu=1}^{d} {\prod_{0 \leqslant k_\nu \leqslant L_\nu - 1}}' \lambda_d(\boldsymbol{k}) \tag{3.2}$$

will be considered as a renormalized critical partition function. The prime in the product in equation (3.2) means that the zero mode term has been excluded.

Here we confine ourselves to the logarithmic finite-size terms in the free energy:

$$F = -\ln \Lambda. \tag{3.3}$$

For simplicity of notation, we assume in the remainder of this section that $L_1 = \ldots = L_d = L$.

As can be seen from the finite-size analysis of Duplantier and David [8], in the case $d = 2$, terms proportional to $\ln L$ arise in equation (3.3) due to cofactors in the partition function Λ which involve $\boldsymbol{k}$-vectors with one non-zero component. These factors are evaluated exactly at any finite $L > 1$, both under periodic and free boundary conditions, with the aid of the identities

$$\prod_{k_1=1}^{L-1} \lambda_{2,2}(k_1, 0) = \prod_{k_1=1}^{L-1} [2 - 2\cos(2\pi k_1/L)] = L^2 \tag{3.4a}$$

$$\prod_{k_1=1}^{L-1} \lambda_{2,0}(k_1, 0) = \prod_{k_1=1}^{L-1} [2 - 2\cos(\pi k_1/L)] = L. \tag{3.4b}$$

Here we consider the general case of a system periodic in d' dimensions and having free boundaries in the remaining $d-d'$ ones. Our derivation is based on the observation that

$$\lambda_{d,d'}(\boldsymbol{k}) = \begin{cases} \lambda_{d-1,d'-1}(k_2,\ldots,k_d)+2-2\cos(2\pi k_1/L) & d'>0 \\ \lambda_{d-1,d'}(k_1,\ldots,k_{d-1})+2-2\cos(\pi k_d/L) & d-d'>0 \end{cases} \tag{3.5}$$

and implements the following generalizations of identities (3.4):

$$\prod_{k=1}^{L-1}[\lambda^2+2-2\cos(2\pi k/L)] = \lambda^{-2}\phi_L^{(0)}(\lambda) \tag{3.6a}$$

$$\prod_{k=1}^{L-1}[\lambda^2+2-2\cos(\pi k/L)] = \lambda^{-1}\phi_L^{(1)}(\lambda) \tag{3.6b}$$

where

$$\begin{aligned} \phi_L^{(0)}(\lambda) &= \{2^{-L}[(\lambda^2+4)^{1/2}+\lambda]^L - 2^L[(\lambda^2+4)^{1/2}+\lambda]^{-L}\}^2 \\ \phi_L^{(1)}(\lambda) &= (\lambda^2+4)^{-1/2}\{2^{-2L}[(\lambda^2+4)^{1/2}+\lambda]^{2L} - 2^{2L}[(\lambda^2+4)^{1/2}+\lambda]^{-2L}\}. \end{aligned} \tag{3.7}$$

Equations (3.6) and (3.7) for any real $\lambda \neq 0$ and integer $L>1$ follow from a known trigonometric identity [13]. Note that, in the limit $\lambda \to 0^+$, identities (3.6a) and (3.6b) reduce to identities (3.4a) and (3.4b), respectively.

Now we start by splitting the free energy (3.3) into the sum

$$\begin{aligned} F_{d,d'} &= -\sum_{\substack{S\subseteq D \\ S\neq\varnothing}} \ln\left(\prod_{\nu\in S}\prod_{k_\nu=1}^{L-1}\prod_{\mu\in D\setminus S}\delta_{k_\mu,0}\lambda_{d,d'}(\boldsymbol{k})\right) \\ &= -\sum_{\substack{S\subseteq D \\ S\neq\varnothing}} \ln\left(\prod_{\nu\in S}\prod_{k_\nu=1}^{L-1}\lambda_{|S|,|S\cap D'|}(\boldsymbol{k})\right) \end{aligned} \tag{3.8}$$

where

$$D=\{1,\ldots,d\} \qquad D'=\begin{cases}\{1,\ldots,d'\}\subseteq D & d'\geqslant 1 \\ \varnothing & d'=0\end{cases} \tag{3.9}$$

and $|S|$ denotes the number of elements in the set S.

It is convenient to introduce the factors

$$Q_{n,p}(L) = \prod_{\nu\in S}\prod_{k_\nu=1}^{L-1}\lambda_{n,p}(\boldsymbol{k}) \tag{3.10}$$

$$R_{n,p}^{(\tau)}(L) = \prod_{\nu\in S}\prod_{k_\nu=1}^{L-1}\phi_L^{(\tau)}(\lambda_{n,p}^{1/2}(\boldsymbol{k})) \qquad \tau=0,1 \tag{3.11}$$

which depend on the integers

$$n=|S| \qquad p=|S\cap D'| \tag{3.12}$$

and on the size L. Then: (i) if $n\geqslant 2$ and $p\geqslant 1$, we use equation (3.6a) with $\lambda=\lambda_{n-1,p-1}^{1/2}(\boldsymbol{k})$ and obtain

$$Q_{n,p}(L) = (Q_{n-1,p-1}(L))^{-1}R_{n-1,p-1}^{(0)}(L) \tag{3.13}$$

(ii) if $n\geqslant 2$ and $p=0$, we use equation (3.6b) with $\lambda=\lambda_{n-1,0}^{1/2}(\boldsymbol{k})$ and obtain

$$Q_{n,0}(L) = (Q_{n-1,0}(L))^{-1/2}R_{n-1,0}^{(1)}(L) \tag{3.14}$$

(iii) if $n=2$ and $p=1$, we have the case of equation (3.4*a*), since

$$Q_{1,1}(L)=\prod_{k_\nu=1}^{L-1}[2-2\cos(2\pi k_\nu/L)]=L^2 \tag{3.15}$$

(iv) if $n=1$ and $p=0$, we have the case of equation (3.4*b*), since

$$Q_{1,0}(L)=\prod_{k_\nu=1}^{L-1}[2-2\cos(\pi k_\nu/L)]=L. \tag{3.16}$$

We can regard now equations (3.13) and (3.14) as recurrent relations with respect to the factors $Q_{n,p}(L)$ and use equations (3.15) and (3.16) as initial conditions.

Thus, if the system is under fully periodic boundary conditions, the repeated use of equation (3.13) for $p=n$ gives

$$Q_{n,n}(L)=L^{2(-1)^{n-1}}\prod_{l=1}^{n-1}(R^{(0)}_{n-l,n-l}(L))^{(-1)^{l-1}} \tag{3.17}$$

and if the system is with fully free boundaries, the repeated use of equation (3.14) gives

$$Q_{n,0}(L)=L^{(-2)^{-n+1}}\prod_{l=1}^{n-1}(R^{(1)}_{n-l,0}(L))^{(-2)^{-l+1}}. \tag{3.18}$$

In the general case one may apply the first p times equation (3.13) and then $(n-p-1)$ times equation (3.14) to obtain

$$Q_{n,p}(L)=L^{(-1)^p(-2)^{-n+p+1}}\prod_{j=1}^{n-p-1}(R^{(1)}_{n-p-j,0}(L))^{(-1)^p(-2)^{-l+1}}\prod_{l=1}^{p}(R^{(0)}_{n-l,n-l}(L))^{(-1)^{l-1}}. \tag{3.19}$$

The crucial observation is that the factors $R^{(\tau)}_{n,p}(L)$ do not contribute terms proportional to $\ln L$ in the free energy (3.8), as can be shown by finite-size analysis extending that of Duplantier and David [8] for $d=2$. Thus, confining ourselves to the logarithmic contributions in the free energy, $\Delta F^{(l)}$, we obtain from equations (3.8) and (3.19)

$$\begin{aligned}\Delta F^{(l)}_{d,d'}&=2\ln L\sum_{\substack{S\subseteq D\\ S\neq\varnothing}}2^{|S\cap D'|}(-\tfrac{1}{2})^{|S|}\\ &=2\ln L\left(\sum_{A\subseteq D'}(-1)^{|A|}\sum_{B\subseteq D\setminus D'}(-\tfrac{1}{2})^{|B|}-1\right)\\ &=\begin{cases}-2\ln L & d'\geq 1\\ -2(1-2^{-d})\ln L & d'=0.\end{cases}\end{aligned} \tag{3.20}$$

A remarkable feature of the above result is that in the absence of corners, i.e. when there is at least one periodic boundary, the logarithmic term does not depend either on the total dimensionality of the system d, or on the number of dimensions with periodic boundaries $d'\geq 1$. On these grounds we define the contribution from the corners of a system with totally free boundaries as

$$\Delta F^{(c)}_{\mathrm{CMD}}=\Delta F^{(l)}_{d,0}-\Delta F^{(l)}_{d,d'>0}=2^{-d+1}\ln L. \tag{3.21}$$

4. Amplitude–exponent relations

Consider two dimers attached to points $m, n\in\mathcal{L}_d$ with coordinates (m_ν, n_ν), $0\leq n_\nu, m_\nu\leq L_\nu-1$, $\nu=1,\ldots,d$ and oriented in a fixed direction e_μ, where e_μ, $1\leq\mu\leq d$, is one of the lattice unit vectors.

The partition function $\Lambda_\mu(m, n)$ for monomer-dimer configurations containing the given pair of dimers can be expressed by a matrix $\mathbf{D}$ as

$$\Lambda_\mu(m, n) = \det(\mathbf{D}+\mathbf{I}). \tag{4.1}$$

The matrix $\mathbf{D}$ differs from $\mathbf{C}$ by the elements $D_{m,m'}$ and $D_{n,n'}$, which are equal to zero if $n' \neq n+e_\mu$ and $m' \neq m+e_\mu$, and by the diagonal elements $D_{n,n} = D_{m,m} = z$. Accordingly, the two-point correlation functions $K_\mu(m, n)$ are given by

$$K_\mu(m, n) = \Lambda_\mu(m, n)/\Lambda(z). \tag{4.2}$$

We start evaluating $K_\mu(m, n)$ in the $d = 2$ case with periodic boundary conditions. Let us put $L_1 = L$, $L_2 \to \infty$ and consider two orientations of dimers: e_1 (figure 2(a)) and e_2 (figure 2(b)) for a pair of dimers separated by the interval Re_2. Evaluating the determinants in equation (4.2) by using Fourier transformations we find the R-dependent part of the correlation functions in the form

$$K_{1,L}(R) = -\left(\frac{1}{2d}\right)^2 Q_{1,L}^2(R) \tag{4.3}$$

where

$$Q_{1,L}(R) = \frac{1}{2\pi L}\sum_{k=0}^{L-1}\int_0^{2\pi} d\alpha \frac{2\cos R\alpha - \cos(R+1)\alpha - \cos(R-1)\alpha}{2-\cos(2\pi k/L)-\cos\alpha} \tag{4.4}$$

and

$$K_{2,L}(R) = \left(\frac{1}{2d}\right)^2 Q_{2,L}^2(R) \tag{4.5}$$

where

$$Q_{2,L}(R) = \frac{1}{\pi L}\sum_{k=0}^{L-1}\int_0^{2\pi} d\alpha \frac{\cos(R+1)\alpha - \cos R\alpha}{2-\cos(2\pi k/L)-\cos\alpha}. \tag{4.6}$$

The integrals over α in equations (4.4) and (4.6) can be evaluated exactly [14], thus yielding

$$Q_{1,L}(R) = -\frac{2}{L}\sum_{k=1}^{L-1}\frac{1-c_L(k)}{1+c_L(k)}\exp(-R\ln c_L(k)) \tag{4.7}$$

and

$$Q_{2,L}(R) = -\frac{1}{L} - \frac{2}{L}\sum_{k=1}^{L-1}(1+c_L(k))^{-1}\exp(-R\ln c_L(k)) \tag{4.8}$$

with

$$\begin{aligned} c_L(k) &= a_L(k) + (a_L^2(k)-1)^{1/2} \\ a_L(k) &= 2-\cos(2\pi k/L). \end{aligned} \tag{4.9}$$

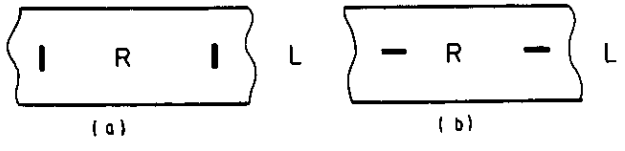

Figure 2. A pair of parallel dimers, separated by R lattice spacings, oriented in (a) the perpendicular and (b) the parallel direction with respect to the axis of an infinitely long block of width L.

Now, it is readily seen that the leading-order contribution in the sums over k when $R \to \infty$ at fixed $L \gg 1$ comes from the terms with $k = 1$ and $k = L-1$, thus giving

$$Q_{1,L}(R) \simeq -4\pi L^{-2} \, \mathrm{e}^{-2\pi R/L} \tag{4.10}$$

and

$$Q_{2,L}(R) \simeq -2L^{-1} - 4L^{-1} \, \mathrm{e}^{-2\pi R/L}. \tag{4.11}$$

Note that the term $-2L^{-1}$ in equation (4.11) gives an R-independent contribution to the dimer–dimer correlation function which should be omitted from the final expression for $K_{2,L}(R)$. The origin of this background term lies in the tree-like nature of the constrained dimer configurations: the connectivity condition implies the existence of a trunk in the tree. The trunk has the form of a self-avoiding walk passing through each cross section perpendicular to the infinitely long direction. Thus, in each such cross section there should be at least one dimer belonging to the trunk which gives a non-vanishing finite-size contribution to the correlation function.

After substitution of equations (4.10) and (4.11) in equations (4.3) and (4.5), respectively, we obtain

$$A_1 = 4\pi \qquad A_2 = 2\pi. \tag{4.12}$$

The bulk correlation functions follow in the limit $L \to \infty$, R fixed, by replacing the sums over k on the right-hand sides of equations (4.7) and (4.8) by the corresponding integrals:

$$\begin{aligned} Q_{1,\infty}(R) &= \frac{1}{\pi} \int_0^{2\pi} \mathrm{d}\alpha \, \frac{1-c(\alpha)}{1+c(\alpha)} \exp(-R \ln c(\alpha)) \\ Q_{2,\infty}(R) &= -\frac{1}{\pi} \int_0^{2\pi} \mathrm{d}\alpha (1+c(\alpha))^{-1} \exp(-R \ln c(\alpha)) \end{aligned} \tag{4.13}$$

where

$$c(\alpha) = 2 - \cos\alpha + [(1-\cos\alpha)(3-\cos\alpha)]^{1/2}.$$

Hence, the leading-order asymptotic behaviour as $R \to \infty$ is

$$\begin{aligned} Q_{1,\infty}(R) &\simeq -(\pi R^2)^{-1} \\ Q_{2,\infty}(R) &\simeq -2(\pi R)^{-1}. \end{aligned} \tag{4.14}$$

Thus, after substitution in equations (4.3) and (4.5), respectively, we obtain

$$x_1 = 2 \qquad x_2 = 1. \tag{4.15}$$

Therefore, the amplitude-exponent relations for the two correlation functions considered agree with equation (1.3):

$$A_i = 2\pi x_i \qquad i = 1, 2. \tag{4.16}$$

Let us turn now to the three-dimensional case. Set $L_1 = L_2 = L$, $L_3 \to \infty$. Consider a pair of dimers oriented in direction $\boldsymbol{e}_1$ and separated by the vector $R\boldsymbol{e}_3$, and a pair of dimers oriented in direction $\boldsymbol{e}_3$ and separated by the vector $R\boldsymbol{e}_3$.

The R-dependent parts of the two correlation functions again take forms (4.4) and (4.5), where in the case of periodic boundary conditions

$$Q_{1,L}(R)=\frac{3}{8\pi L^2}\sum_{k_1=0}^{L-1}\sum_{k_2=0}^{L-1}\int_0^{2\pi}\mathrm{d}\alpha\,\frac{2\cos R\alpha-\cos(R+1)\alpha-\cos(R-1)\alpha}{1+\lambda_{2,2}(\boldsymbol{k})/2-\cos\alpha}$$
$$Q_{2,L}(R)=\frac{3}{2\pi L^2}\sum_{k_1=0}^{L-1}\sum_{k_2=0}^{L-1}\int_0^{2\pi}\mathrm{d}\alpha\,\frac{\cos(R+1)\alpha-\cos R\alpha}{1+\lambda_{2,2}(\boldsymbol{k})/2-\cos\alpha}. \tag{4.17}$$

Again the integration over α can be performed exactly, the result being

$$Q_{1,L}(R)=\frac{3}{2L^2}\sum_{k_1=0}^{L-1}{\sum_{k_2=0}^{L-1}}'\frac{1-c_L(\boldsymbol{k})}{1+c_L(\boldsymbol{k})}\exp(-R\ln c_L(\boldsymbol{k}))$$
$$Q_{2,L}(R)=-\frac{3}{L^2}-\frac{6}{L^2}\sum_{k_1=0}^{L-1}{\sum_{k_2=0}^{L-1}}'[1+c_L(\boldsymbol{k})]^{-1}\exp(-R\ln c_L(\boldsymbol{k})) \tag{4.18}$$

where the prime indicates that the term with $\boldsymbol{k}=0$ has been omitted from the sum, and

$$c_L(\boldsymbol{k})=a_L(\boldsymbol{k})+(a_L^2(\boldsymbol{k})-1)^{1/2}$$
$$a_L(\boldsymbol{k})=3-\cos(2\pi k_1/L)-\cos(2\pi k_2/L). \tag{4.19}$$

Hence, the leading-order asymptotic form when $R\to\infty$ at fixed $L\gg 1$ is given by the terms with $\boldsymbol{k}=(0,1)$, $(0,L-1)$, $(1,0)$ and $(L-1,0)$:

$$Q_{1,L}(R)\simeq -6\pi L^{-3}\,\mathrm{e}^{-2\pi R/L}$$
$$Q_{2,L}(R)\simeq -3L^{-2}-12L^{-2}\,\mathrm{e}^{-2\pi R/L}. \tag{4.20}$$

Therefore, after substitution in equations (4.3) and (4.5), we obtain $A_1=4\pi$ and $A_2=2\pi$.

In the bulk limit $L\to\infty$ one obtains from equations (4.18)

$$Q_{1,\infty}(R)=\frac{3}{8\pi^2}\int_0^{2\pi}\mathrm{d}\alpha\int_0^{2\pi}\mathrm{d}\beta\,\frac{1-c(\alpha,\beta)}{1+c(\alpha,\beta)}\exp(-R\ln c(\alpha,\beta))$$
$$Q_{2,\infty}(R)=-\frac{3}{2\pi^2}\int_0^{2\pi}\mathrm{d}\alpha\int_0^{2\pi}\mathrm{d}\beta(1+c(\alpha,\beta))^{-1}\exp(-R\ln c(\alpha,\beta)) \tag{4.21}$$

where

$$c(\alpha,\beta)=3-\cos\alpha-\cos\beta+[(2-\cos\alpha-\cos\beta)(4-\cos\alpha-\cos\beta)]^{1/2}. \tag{4.22}$$

Therefore, the leading-order asymptotic form as $R\to\infty$ is

$$Q_{1,\infty}(R)\simeq -3(\pi R^3)^{-1}$$
$$Q_{2,\infty}(R)\simeq -6(\pi R^2)^{-1} \tag{4.23}$$

and from equations (4.3), (4.5) and (4.23) it follows that $x_1=3$ and $x_2=2$.

Hence, the amplitude-exponent relations

$$A_1=(4/3)\pi x_1 \qquad A_2=\pi x_2 \tag{4.24}$$

are in contradiction to equation (1.11).

Henkel [10] noticed a breakdown of relation (1.3) in the $d=3$ case under periodic boundary conditions and suggested that the relation is restored under antiperiodic boundary conditions. For the model considered here, the antiperiodic conditions in

directions e_1 and e_2 imply sign inversion of the matrix elements $C_{i,j}$ which connect vertices on opposite edges of $\mathscr{L}_3$ oriented in the given directions. In this case, instead of the pure tree-like problem, we obtain a model containing a number of closed loops of period L_1 or L_2 with tree-like branches attached to them.

For antiperiodic boundary conditions in the two finite dimensions we have

$$Q_{1,L}(R)=\frac{3}{8\pi L^2}\sum_{k_1=0}^{L-1}\sum_{k_2=0}^{L-1}\int_0^{2\pi}\mathrm{d}\alpha\,\frac{2\cos R\alpha-\cos(R+1)\alpha-\cos(R-1)\alpha}{1+\bar{\lambda}_{2,0}(\boldsymbol{k})/2-\cos\alpha}$$
$$Q_{2,L}(R)=\frac{3}{2\pi L^2}\sum_{k_1=0}^{L-1}\sum_{k_2=0}^{L-1}\int_0^{2\pi}\mathrm{d}\alpha\,\frac{\cos(R+1)\alpha-\cos R\alpha}{1+\bar{\lambda}_{2,0}(\boldsymbol{k})/2-\cos\alpha}. \tag{4.25}$$

Proceeding in complete analogy with the case of periodic boundary conditions, we obtain after the integration over α

$$Q_{1,L}(R)=\frac{3}{2L^2}\sum_{k_1=0}^{L-1}\sum_{k_2=0}^{L-1}\frac{1-\bar{c}_L(\boldsymbol{k})}{1+\bar{c}_L(\boldsymbol{k})}\exp(-R\ln\bar{c}_L(\boldsymbol{k}))$$
$$Q_{2,L}(R)=-\frac{6}{L^2}\sum_{k_1=0}^{L-1}\sum_{k_2=0}^{L-1}[1+\bar{c}_L(\boldsymbol{k})]^{-1}\exp(-R\ln\bar{c}_L(\boldsymbol{k})) \tag{4.26}$$

where

$$\bar{a}_L(\boldsymbol{k})=3-\cos[\pi(2k_1+1)/L]-\cos[\pi(2k_2+1)/L]$$
$$\bar{c}_L(\boldsymbol{k})=\bar{a}_L(\boldsymbol{k})+(\bar{a}_L^2(\boldsymbol{k})-1)^{1/2}. \tag{4.27}$$

Hence, in the regime $R\to\infty$ at fixed $L\gg 1$ we obtain up to leading order

$$Q_{1,L}(R)\simeq 3\pi 2^{1/2}L^{-3}\,\mathrm{e}^{-2^{1/2}\pi R/L}$$
$$Q_{2,L}(R)\simeq -12L^{-2}\,\mathrm{e}^{-2^{1/2}\pi R/L}. \tag{4.28}$$

Therefore, the substitution in equations (4.3) and (4.5) gives $A_1=A_2=2^{1/2}\pi$.

Thus, we obtain now the amplitude–exponent relations

$$A_1=\frac{2^{3/2}}{3}\pi x_1 \qquad A_2=\frac{2^{1/2}}{3}\pi x_2 \tag{4.29}$$

which are again in contradiction to equation (1.11).

5. Discussion

We have presented an exactly solvable model, the CMD model, which is critical in the limit of infinite dimer activity, when it coincides with the ST model considered in two dimensions by Duplantier and David [8]. The CMD model permits an exact solution for the logarithmic finite-size corrections in the free energy at arbitrary dimensionality d.

As mentioned in the introduction, the only other model for which analogous results are available is the Gaussian model in the near-critical regime $mL\to 0$ [7]. By comparing equations (1.4) and (1.6) with equation (3.20) we see that the universal amplitudes of the $\ln L$ terms in the two models differ by the constant factor -2 for any d, both under periodic and free boundary conditions. However, the free energy of the Gaussian model diverges in the zero-mass limit $m\to 0$ unless the zero-eigenvalue mode is removed

manually. The above proportionality of the universal amplitudes is due to the fact that the partition function of the *critical* Gaussian model with the zero-eigenvalue mode omitted,

$$Z'_{\mathrm{G}} = (2\pi)^{(N-1)/2}\left(\prod_{\nu=1}^{d}\prod_{0\leqslant k_\nu\leqslant L_\nu-1}{}' \lambda_d(\boldsymbol{k})\right)^{-1/2}$$

is simply related to the partition function of the ST model Λ (see equation (3.2)), namely

$$\Lambda = (2\pi)^{(N-1)}(Z'_{\mathrm{G}})^{-2}.$$

It should be emphasized, however, that in the CMD model the exclusion of the zero-eigenvalue model takes place naturally, in the process of solution by the Kirhhoff theorem.

In conclusion we note that in the presented exactly solvable model the amplitude-exponent relation holds in two dimensions and breaks down in three dimensions, under both periodic and antiperiodic boundary conditions.

A possible explanation of this breakdown consists in the different role the antiperiodic boundary conditions play in the spin models considered by Henkel and in our combinatorial problem [14]. In the case of the spherical and Ising models, the antiperiodic boundary conditions create a real interface between domains of spins with opposite direction. In the more general situation of the Potts model, an interface between domains with different states may be created by similar rotation of variables. In contrast, the CMD model considered here is a tree-like graph model which we do not know to have any reasonable spin representation. In this case antiperiodicity implies merely the appearance of closed loops in the graph; the latter cannot be considered as interfaces between domains of different phases.

Presumably, the consideration of open boundary conditions in the CMD model and a comparison with analogous results for spin models would answer the question about the status of the amplitude-exponent relation in three and higher dimensions.

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